Discrete-Core Nonemptiness for Approval Elections with at Most Eight Voters

Chenghan Zhou

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Abstract

We prove, by an exact computer-assisted argument, that every approval-based committee election with at most eight equally weighted voters has a committee in the discrete core. The cases of at most five voters were previously known; the new cases are six, seven, and eight voters. The proof begins with a fractional Nash-core allocation and its proportional-payment prices. After choosing such an allocation with minimum fractional support, every unresolved instance reduces to a full-column-rank binary antichain with at most as many columns as voters. The remaining obstruction is a rational floor region of this small residual matrix.

The three new bounded-voter steps use successively stronger rounding criteria. For six voters, every unresolved residual floor region has a usable tight row. For seven voters, every remaining region admits an α-separated one-deficit rounding. For eight voters, ordinary α-separation is no longer sufficient; we derive an individual-saturation lower bound depending only on residual data and combine it with a coalition price inequality. This yields fixed and adaptive one-deficit price–saturation certificates. Exhaustive canonical enumeration and exact rational verification certify every residual floor region. In the eight-row computation there are 13,269,620 positive-dual matrices and 1,085,483 unresolved regions; 1,085,459 receive fixed certificates and 24 receive adaptive certificates, with no failures. The companion artifact, available at `https://github.com/chenghanzh/discrete-core-at-most-eight-artifact`, contains the source, proof-relevant records, exact replay programs, manifests, and reproduction scripts.

1 Reduction to finite residual problems

An approval-based committee election consists of a finite set of voters, a finite set of candidates, and an approval set for each voter. A coalition may object to a committee if it can afford a smaller committee, in proportion to its size, that every coalition member strictly prefers. A committee against which no such objection exists lies in the discrete core. Discrete-core nonemptiness is a strong proportional-representation guarantee and remains open for unrestricted elections.

Several complementary parameterized results are known. Core committees exist when the committee size is at most eight and when the number of candidates is at most fifteen [4]. Nonemptiness for at most five voters and, more generally, for at most five distinct approval types with arbitrary multiplicities was proved in [3]; the affine-monoid rounding argument used there breaks at six voters. On the fractional side, the fractional Nash core was introduced in [1], together with its connection to proportional payments and the fractional core. The regular-case existence statement needed here is proved directly in Theorem 1.6, using the capped perspective-entropy program of [2].

Our main result is the following.

Theorem 1.1. *Every approval-based committee election with at most eight equally weighted voters has a committee in the discrete core.*

The theorem places no restriction on the committee budget or on the number of candidates. Its genuinely new coverage is complementary to the small-budget, small-candidate, and at-most-five-type theorems above.

Voters versus voter types. The parameter in Theorem 1.1 is the total number of individual, equally weighted voters. It is not the number of distinct approval types with arbitrary multiplicities. The latter is a weighted problem in which coalition entitlements depend on type multiplicities; the finite certificates in this note do not quantify over those additional weights.

Proof architecture. The main proof is organized into two sections. Section 1 develops the common reduction: after constructing a fractional Nash-core pair and choosing one with minimum fractional support, every unresolved election is reduced to a full-column-rank binary antichain with at most n columns, an integer residual budget, and a rational utility-floor region. Section 2 establishes the common blocker inequalities and one-deficit rounding principles, then treats six, seven, and eight voters using successively stronger criteria: a tight-row argument, α-separation, and price–saturation certificates. The exact finite classifications and their verification are collected in Appendix A, and the proof of Theorem 1.1 is completed at the end of the eight-voter case.

The computation does not enumerate full elections or the candidates outside the fractional set. It enumerates only the small antichain of fractional candidate supports and its rational utility-floor regions. The Nash prices summarize the omitted fully selected and unselected candidates.

1.1 Model, core notions, and regularity

Let $N = [n] = \{1, \ldots, n\}$ be the voters and let C be the candidates. Voter i approves a set $\mathcal{A}_i \subseteq C$. Candidate c has support $N(c) := \{i \in N : c \in \mathcal{A}_i\}$. For $x \in [0,1]^C$, define $|x| := \sum_{c \in C} x_c$ and $u_i(x) := \sum_{c \in \mathcal{A}_i} x_c$. An integral committee is a vector in $\{0,1\}^C$, identified with its support. Fix an integer $K \geq 1$. We allow committees of size at most K; a smaller committee may be padded with unused candidates because adding candidates cannot lower any utility.

Definition 1.2 (Discrete and fractional core)**.** An integral committee T *blocks* an integral committee W through a nonempty coalition $S \subseteq N$ if $|T| \leq K|S|/n$ and $u_i(T) > u_i(W)$ for every $i \in S$. An integral committee W, $|W| \leq K$, lies in the *discrete core* if no such T and S exist.

A fractional committee $y \in [0,1]^C$ *blocks* a fractional committee $x \in [0,1]^C$ through a nonempty coalition $S \subseteq N$ if $|y| \leq K|S|/n$ and $u_i(y) > u_i(x)$ for every $i \in S$. A fractional committee x, $|x| \leq K$, lies in the *fractional core* if no such y and S exist.

The following criterion transfers a fractional core to a discrete core.

Lemma 1.3 (Floor rounding)**.** *Let x be in the fractional core. If an integral committee W satisfies $|W| \leq K$ and $u_i(W) \geq \lfloor u_i(x) \rfloor$ for every $i \in N$, then W is in the discrete core.*

Proof. Suppose, for contradiction, that an integral committee T blocks W through a coalition S. For every $i \in S$, integrality gives $u_i(T) \geq u_i(W) + 1 \geq \lfloor u_i(x) \rfloor + 1 > u_i(x)$. Because $\{0,1\}^C \subseteq [0,1]^C$, the committee T is also a fractional committee with the same cost. Hence T blocks x through S, contradicting that x lies in the fractional core. $\square$

Following Assumption 1 in [1], we use the following definition.

Definition 1.4 (Regular election)**.** An election is *regular* if $|\bigcup_{i \in S} \mathcal{A}_i| > K|S|/n$ for every nonempty proper coalition $S \subsetneq N$.

The regular case is the main case of the proof. We first show a lemma on irregular case by reducing it to an election with fewer voters.

Lemma 1.5 (Irregular-coalition reduction)**.** *Let $S \subsetneq N$ be a nonempty proper coalition, and define $U := \bigcup_{i\in S} \mathcal{A}_i$. Suppose $|U| \le K|S|/n$. Put $N' := N \setminus S$, $C' := C \setminus U$, and $K' := K - |U|$. If the residual election (N', C', K') has a core committee W', then $U \cup W'$ is a core committee of the original election.*

Proof. Every voter in S receives every candidate she approves and therefore cannot strictly improve. Moreover, $K'/|N'| = (K - |U|)/(n - |S|) \ge K/n$. Suppose a coalition $Q \subseteq N'$ blocks $U \cup W'$ through an integral committee T. Put $T' := T \setminus U$. Since U and W' are disjoint, for every $i \in Q$,

$$u_i(T') - u_i(W') = u_i(T) - u_i(U \cup W') + u_i(U) - u_i(T \cap U) > 0.$$

Also, $|T'| \le |T| \le K|Q|/n \le K'|Q|/|N'|$. Hence T' blocks W' in the residual election, a contradiction. $\square$

1.2 Fractional Nash-core prices

Let $C^+ := \bigcup_{i\in N} \mathcal{A}_i$ be the candidates approved by at least one voter. Candidates outside C^+ may be deleted, and if $|C^+| \le K$, then selecting all of C^+ is core-stable. Thus we focus on the case $|C^+| > K$. Following [1], a pair satisfying the weight-selection coupling and uncapped Nash-optimality conditions in items (i)–(ii) below is called a fractional Nash-core pair.

Theorem 1.6 (Regular fractional Nash-core)**.** *Suppose the election is regular and $|C^+| > K$. Then $\mathcal{A}_i \neq \varnothing$ for every $i \in N$, and there are vectors $w \in (0,1]^C$ and $x \in [0,1]^C$ with $|x| = K$ and the following properties.*

(i) If $w_c < 1$, then $x_c = 1$.

(ii) The allocation x maximizes $\sum_{i\in N} \log\left(\sum_{c\in\mathcal{A}_i} w_c z_c\right)$ over the uncapped budget simplex $\{z \in \mathbb{R}^C_{\ge 0} : |z| \le K\}$.

(iii) Every weighted voter utility is positive.

(iv) The allocation x lies in the fractional core.

(v) If $t := K/n$, $\phi_i := \sum_{c\in\mathcal{A}_i} w_c x_c$, and $\alpha_i := t/\phi_i$, and if $\alpha(R) := \sum_{i\in R} \alpha_i$, then

$$w_c\alpha(N(c)) = 1 \quad \forall c \in C \text{ with } x_c > 0, \qquad w_c\alpha(N(c)) \le 1 \quad \forall c \in C \text{ with } x_c = 0.$$

Proof. If $n > 1$, regularity applied to $S = \{i\}$ gives $|\mathcal{A}_i| > K/n > 0$ for every $i \in N$; if $n = 1$, the conclusion follows from $\mathcal{A}_1 = C^+$ and $|C^+| > K$. Work first on the approved candidates C^+. Put $t := K/n$ and, for $\varnothing \neq S \subseteq N$, write $\mathcal{A}(S) := \bigcup_{i\in S} \mathcal{A}_i$. Regularity gives $t|S| < |\mathcal{A}(S)|$ for every proper nonempty S, while $|C^+| > K = tn$ gives the same inequality for $S = N$. Hence

$$\rho := \max_{\varnothing\neq S\subseteq N} \frac{t|S|}{|\mathcal{A}(S)|} < 1.$$

Choose q with $\rho < q < 1$. The max-flow/min-cut theorem applied to the approval graph gives numbers $b^0_{ic} \ge 0$, supported on approval edges, such that

$$\sum_{c\in\mathcal{A}_i} b^0_{ic} = t \quad \forall i \in N, \qquad \sum_{i:c\in\mathcal{A}_i} b^0_{ic} \le q \quad \forall c \in C^+.$$

Indeed, the cut condition for a voter set S is precisely $t|S| \le q|\mathcal{A}(S)|$. By mixing b^0 with a sufficiently small positive row-wise distribution on every approval edge, we obtain a feasible array with every approval-edge entry positive and every column sum strictly below one.

For an array $b = (b_{ic})$ supported on approval edges, set

$$x_c(b) := \sum_{i: c\in\mathcal{A}_i} b_{ic}.$$

Consider the approval-specialized capped perspective-entropy program

$$\begin{aligned} \text{minimize} \quad & G(b) := \sum_{c\in C^+} \sum_{i: c\in\mathcal{A}_i} b_{ic} \log \frac{b_{ic}}{x_c(b)} && \\ \text{subject to} \quad & \sum_{c\in\mathcal{A}_i} b_{ic} = t && \forall i \in N, \\ & x_c(b) \le 1 && \forall c \in C^+, \\ & b_{ic} \ge 0 && \forall i \in N,\ \forall c \in \mathcal{A}_i. \end{aligned} \tag{1}$$

We use the continuous convention $0\log(0/0) = 0$. The objective is convex, and its column-wise subdifferential is the perspective-entropy subdifferential computed in [2, Lemmas 5 and 6]. The feasible set is compact, so an optimum b exists. The strictly feasible array above is a point of continuity of G and gives Slater's condition. Strong duality therefore yields row multipliers $\lambda_i \in \mathbb{R}$ and cap multipliers $\mu_c \ge 0$ such that b minimizes, over all nonnegative arrays, the Lagrangian

$$\mathcal{L}(b;\lambda,\mu) := G(b) - \sum_{i\in N} \lambda_i \left(\sum_{c\in\mathcal{A}_i} b_{ic} - t \right) + \sum_{c\in C^+} \mu_c (x_c(b) - 1),$$

and $\mu_c(x_c(b) - 1) = 0$ for every $c \in C^+$.

Let $x_c := x_c(b)$. Suppose that $b_{ic} = 0 < x_c$ for some approval edge (i, c). Let d^{ic} be the array with a one in coordinate (i, c) and zeros elsewhere. The one-sided directional derivative of the perspective-entropy objective is

$$\lim_{\varepsilon\downarrow 0} \frac{G(b + \varepsilon d^{ic}) - G(b)}{\varepsilon} = -\infty.$$

The remaining terms of the Lagrangian have the finite directional derivative $-\lambda_i + \mu_c$. Hence $\mathcal{L}(b+\varepsilon d^{ic};\lambda,\mu) < \mathcal{L}(b;\lambda,\mu)$ for all sufficiently small $\varepsilon > 0$, contradicting the minimizing property of b. Therefore $b_{ic} > 0$ for every $i \in N(c)$ whenever $x_c > 0$.

The subdifferential characterization now implies that $\partial G(b)$ is nonempty. First-order optimality of b for the Lagrangian over the nonnegative orthant yields nonnegativity multipliers $\eta_{ic} \ge 0$ and a subgradient $g \in \partial G(b)$ such that

$$g_{ic} - \lambda_i + \mu_c - \eta_{ic} = 0, \qquad \mu_c(x_c - 1) = 0, \qquad \eta_{ic} b_{ic} = 0. \tag{2}$$

If $x_c > 0$, the subgradient description gives $g_{ic} = \log(b_{ic}/x_c)$. If $x_c = 0$, it gives

$$\sum_{i\in N(c)} e^{g_{ic}} \le 1. \tag{3}$$

Define

$$\alpha_i := e^{\lambda_i} > 0, \qquad w_c := e^{-\mu_c} \in (0, 1] \quad \forall c \in C^+, \qquad \alpha(R) := \sum_{i\in R} \alpha_i.$$

For $x_c > 0$, complementary slackness in (2) gives $\eta_{ic} = 0$, and hence $b_{ic} = \alpha_i w_c x_c$. Summing over $i \in N(c)$ yields

$$w_c \alpha(N(c)) = 1 \quad \forall c \in C^+ \text{ with } x_c > 0. \tag{4}$$

For $x_c = 0$, complementary slackness gives $\mu_c = 0$, so $w_c = 1$. Equation (2) gives $g_{ic} = \lambda_i + \eta_{ic} \ge \lambda_i$, and (3) therefore yields

$$\alpha(N(c)) \le 1 \quad \forall c \in C^+ \text{ with } x_c = 0. \tag{5}$$

Moreover, $w_c < 1$ implies $\mu_c > 0$ and therefore $x_c = 1$. Summing the row constraints gives $|x| = nt = K$. For every voter i,

$$t = \sum_{c \in \mathcal{A}_i} b_{ic} = \alpha_i \sum_{c \in \mathcal{A}_i} w_c x_c,$$

so the weighted utility $\phi_i := \sum_{c \in \mathcal{A}_i} w_c x_c$ is positive and $\alpha_i = t/\phi_i$.

Restore every candidate in $C \setminus C^+$ by setting $x_c = 0$ and $w_c = 1$. The price relations remain valid. To prove Nash optimality, let

$$F(z) := \sum_{i \in N} \log \left(\sum_{c \in \mathcal{A}_i} w_c z_c \right).$$

At x,

$$\frac{\partial F}{\partial z_c}(x) = w_c \sum_{i \in N(c)} \frac{1}{\phi_i} = \frac{w_c}{t} \alpha(N(c)).$$

By (4)–(5), this derivative equals $1/t$ when $x_c > 0$ and is at most $1/t$ when $x_c = 0$. Together with $|x| = K$, these are the KKT conditions for maximizing the concave function F over $z \geq 0$, $|z| \leq K$. This proves item (ii).

It remains to prove fractional-core stability. Suppose that a fractional committee y blocks x through a nonempty coalition S. Let $D := \{c : w_c < 1\}$. Since $x_c = 1$ on D and $0 \leq y_c \leq 1$, for every $i \in S$,

$$\begin{aligned} \sum_{c \in \mathcal{A}_i} w_c (y_c - x_c) &= \sum_{c \in \mathcal{A}_i \cap D} w_c (y_c - 1) + \sum_{c \in \mathcal{A}_i \setminus D} (y_c - x_c) \\ &\geq u_i(y) - u_i(x) > 0. \end{aligned}$$

Thus the weighted utility of every voter in S also strictly increases. Since $|y| > 0$, the vector $\widehat{y} := (K/|y|)y$ is feasible for the uncapped comparison problem. Furthermore,

$$y^\top \nabla F(x) = \sum_{i \in N} \frac{\sum_{c \in \mathcal{A}_i} w_c y_c}{\phi_i} > |S|, \qquad x^\top \nabla F(x) = n.$$

The blocking budget bound $|y| \leq K|S|/n$ gives $K/|y| \geq n/|S|$, and hence

$$(\widehat{y} - x)^\top \nabla F(x) > 0,$$

contradicting first-order optimality of x. Therefore x lies in the fractional core. All five stated properties follow. □

The fractional-core argument in the proof used only weight-selection coupling and uncapped Nash optimality; consequently, in the present regular branch, the allocation of every fractional Nash-core pair lies in the fractional core.

The comparison problem in item (ii) is uncapped: it does not impose $z_c \leq 1$. The selected allocation x itself is capped. This distinction is used in the budget-saturation and minimum-fractionality arguments.

Let (w, x) be any fractional Nash-core pair in the regular branch. Set $t := K/n$, $\phi_i := \sum_{c \in \mathcal{A}_i} w_c x_c$, and $\alpha_i := t/\phi_i > 0$. Partition the candidates into

$$I := \{c : x_c = 1\}, \quad F := \{c : 0 < x_c < 1\}, \quad O := \{c : x_c = 0\}.$$

For $R \subseteq N$, write $\alpha(R) := \sum_{i \in R} \alpha_i$.

Proposition 1.7 (Nash-price consequences)**.** *For every fractional Nash-core pair (w, x) in the regular branch:*

(i) for every candidate c,

$$\alpha(N(c)) = 1 \ \forall c \in F, \quad \alpha(N(c)) \le 1 \ \forall c \in O, \quad w_c\alpha(N(c)) = 1 \ \forall c \in I;$$

(ii) candidates approved by no voter may be deleted, and if at most K candidates are approved by at least one voter, selecting all of them is core-stable;

(iii) if more than K candidates are approved by at least one voter, then every fractional Nash-core allocation exhausts the budget: $|x| = K$.

Proof. Part (ii) is immediate. For part (iii), suppose $|x| < K$. Since more than K candidates are approved by at least one voter, some approved candidate c satisfies $x_c < 1$. Increasing this coordinate slightly preserves budget feasibility, strictly raises at least one weighted voter utility, and lowers none, contradicting Nash optimality in the uncapped comparison problem. Hence $|x| = K$.

Every weighted voter utility is positive: the uncapped comparison problem has a feasible point giving every voter positive utility, whereas a zero utility would make the logarithmic objective equal to $-\infty$. Let λ be the KKT multiplier of the budget constraint in the uncapped Nash problem. For every candidate c,

$$w_c \sum_{i\in N(c)} \frac{1}{\phi_i} \le \lambda,$$

with equality whenever $x_c > 0$. Multiplying the equalities by x_c, summing over positive coordinates, and using $|x| = K$ gives

$$\lambda K = \sum_{c\in C} x_c w_c \sum_{i\in N(c)} \frac{1}{\phi_i} = \sum_{i\in N} \frac{\phi_i}{\phi_i} = n.$$

Thus $\lambda = n/K = 1/t$, and therefore

$$w_c\alpha(N(c)) = 1 \quad \forall c \in C \text{ with } x_c > 0, \qquad w_c\alpha(N(c)) \le 1 \quad \forall c \in C \text{ with } x_c = 0.$$

For $c \in F \cup O$, weight-selection coupling implies $w_c = 1$, proving part (i). □

1.3 Minimum-fractionality and the residual matrix

In the regular branch fixed above, among all fractional Nash-core pairs choose one whose allocation has the minimum possible number of fractional coordinates. Such a choice exists because the number of fractional coordinates is an integer between 0 and $|C|$.

Recall that $F := \{c \in C : 0 < x_c < 1\}$ is the set of fractional candidates. Let $m := |F|$ and identify the elements of F with $[m]$. Let $A \in \{0,1\}^{n\times m}$ be the voter-by-fractional-candidate approval matrix, so $A_{ic} = 1$ iff $c \in \mathcal{A}_i$. Let $f := x_F \in (0,1)^m$ be the vector of fractional selection weights. By Proposition 1.7(iii), $\kappa := \mathbf{1}^\top f = K - |I| \in \mathbb{Z}_{\ge 0}$ is the residual budget assigned to the fractional candidates. Let $r := Af$ be the vector of utilities contributed by the fractional candidates. Write $r = h + \delta$, where $h := \lfloor r \rfloor = \lfloor Af \rfloor \in \mathbb{Z}^n_{\ge 0}$ is the componentwise integer part of r, and $\delta := r - h$ is its fractional remainder, with $0 \le \delta_i < 1$. Let $b_i := |I \cap \mathcal{A}_i|$ be the number of fully selected candidates approved by voter i, and let $\beta_i := \sum_{c\in I\cap\mathcal{A}_i} w_c$ be their contribution to voter i's utility weighted by w_c in Theorem 1.6. Since every fractional candidate has weight one,

$$\alpha_i(\beta_i + r_i) = t, \quad \beta_i = \frac{t}{\alpha_i} - r_i, \quad b_i \ge \beta_i. \tag{6}$$

For $z \in \{0,1\}^m$, we call Az its residual utility vector. An integer vector h is *implementable* at budget κ if there exists $z \in \{0,1\}^m$ with $\mathbf{1}^\top z \le \kappa$ and $Az \ge h$. Since $\kappa \le m$, such a selection may be padded to exactly κ columns without decreasing utility.

If h is implementable, choose such a vector z and set $W := I \cup \operatorname{supp}(z)$. Then $|W| = |I| + \mathbf{1}^\top z \le |I| + \kappa = K$, and every voter i receives $u_i(W) = b_i + (Az)_i \ge b_i + h_i = \lfloor b_i + r_i \rfloor = \lfloor u_i(x) \rfloor$. Since x lies in the fractional core, Lemma 1.3 implies that W lies in the discrete core. Thus, we focus on the case where h is not implementable for the rest of the proof.

Proposition 1.8 (Minimum-fractionality reduction)**.** *If the residual floor h is not implementable, then:*

(i) the columns of A are pairwise distinct and form an antichain;
(ii) A has full column rank, hence $m \le n$;
(iii) $2 \le \kappa \le m-2$, hence $4 \le m \le n$;
(iv) $A^\top \alpha = \mathbf{1}$ for some $\alpha \in \mathbb{R}^n_{>0}$.

Proof. (i). If two fractional candidates have identical supports, transfer mass from one coordinate to the other while preserving their sum until one coordinate reaches 0 or 1. Every weighted voter utility and the total budget remain unchanged, so the new pair is Nash-optimal and satisfies the weight-selection coupling, but has fewer fractional coordinates.

If fractional candidates a, b satisfy $N(a) \subsetneq N(b)$, then for sufficiently small $\varepsilon > 0$ the changes $f_a \leftarrow f_a - \varepsilon$ and $f_b \leftarrow f_b + \varepsilon$ preserve the budget, change no utility outside $N(b) \setminus N(a)$, and strictly raise the utility of every voter in that nonempty set. The Nash objective therefore increases, a contradiction.

(ii). Suppose $A\gamma = 0$ for nonzero $\gamma \in \mathbb{R}^m$, oriented so that $\mathbf{1}^\top \gamma \le 0$. Since $f \in (0,1)^m$, $\varepsilon^* := \sup\{\varepsilon \ge 0 : f + \varepsilon\gamma \in [0,1]^m\}$ is positive and finite, and at least one coordinate of $f + \varepsilon^*\gamma$ is integral. All voter utilities are unchanged. If $\mathbf{1}^\top\gamma = 0$, the total budget is unchanged and the fractional support shrinks. If $\mathbf{1}^\top\gamma < 0$, the objective is unchanged but the resulting Nash optimum has slack budget, contradicting Proposition 1.7(iii). Thus A has full column rank.

(iii). The cases $\kappa = 0$ and $\kappa = m$ are integral. If $\kappa = 1$, then every row with positive floor must approve every fractional column, so any one column implements all positive floor coordinates. If $\kappa = m-1$, put $g := \mathbf{1} - f$. Then $g > 0$ and $\mathbf{1}^\top g = 1$. A row of degree zero has $r_i = h_i = 0$. A row of degree $1 \le d_i < m$ has $r_i = d_i - (Ag)_i \in (d_i - 1, d_i)$, so $h_i = d_i - 1$, while a row of degree m has $r_i = h_i = m-1$. Selecting all but any one column therefore dominates every row floor. Nonimplementability consequently forces $2 \le \kappa \le m-2$, and hence $m \ge 4$.

(iv). For every fractional candidate c, Proposition 1.7(i) gives $\alpha(N(c)) = 1$. These equations are exactly $A^\top\alpha = \mathbf{1}$, with $\alpha > 0$ by construction. □

For a fixed matrix A, define its positive-dual domain by $\mathcal{D}(A) := \{\alpha \in \mathbb{R}^n_{>0} : A^\top\alpha = \mathbf{1}\}$. The computer-assisted search later enumerates only full-column-rank antichains with $4 \le m \le n$ and $2 \le \kappa \le m-2$ for which $\mathcal{D}(A) \ne \varnothing$.

Floor-Rounding. An instance of FLOOR-ROUNDING consists of an approval election $\mathcal{E} = (N, C, (\mathcal{A}_i)_{i\in N}, K)$ and a rational fractional committee $x \in [0,1]^C$ with $|x| \le K$. The question is whether there exists an integral committee $W \subseteq C$ such that

$$|W| \le K \qquad \text{and} \qquad u_i(W) \ge \lfloor u_i(x) \rfloor \quad \text{for every } i \in N.$$

The next proposition shows that the residual matrix conditions above do not, by themselves, make arbitrary floor-preserving rounding computationally tractable.

Proposition 1.9 (NP-completeness of floor-preserving rounding)**.** *The problem FLOOR-ROUNDING is NP-complete.*

Moreover, its NP-hardness persists even on instances satisfying all of the following properties, for some integer $q \ge 2$:

1. $n = 6q$, $|C| = 3q$, and $K = q$;

2. $x_c = \frac{1}{3}$ *for every* $c \in C$*, so every candidate is fractional and the residual budget is* $\kappa = \mathbf{1}^\top x = q$*;*
3. *the election is regular;*
4. *with the all-one weight vector* $w = \mathbf{1}$*, the pair* (w, x) *is a fractional Nash-core pair, and* x *lies in the fractional core;*
5. *the voter-by-fractional-candidate matrix* $A \in \{0,1\}^{6q \times 3q}$ *has full column rank, its columns are pairwise distinct and form an antichain, and*

$$\mathcal{D}(A) = \{\alpha \in \mathbb{R}^{6q}_{>0} : A^\top \alpha = \mathbf{1}\} \neq \varnothing;$$

6. $2 \leq \kappa \leq |C| - 2$.

Thus, deciding whether a supplied fractional core admits a floor-preserving integral rounding remains NP-hard even under the matrix-level structural conclusions used in Proposition 1.8.

Proof. Membership in NP is immediate. An integral committee $W \subseteq C$ is a polynomial-size certificate. Its cardinality and all inequalities $|W \cap \mathcal{A}_i| \geq \left\lfloor \sum_{c \in \mathcal{A}_i} x_c \right\rfloor$ can be checked in polynomial time using exact integer and rational arithmetic.

For NP-hardness, we reduce from *Restricted Exact Cover by 3-Sets* (RXC3), which is NP-complete [8, Appendix A]. An instance consists of a universe U, with $|U| = 3q$, and a collection $\mathcal{T} \subseteq \binom{U}{3}$ such that every element of U belongs to exactly three triples in $\mathcal{T}$. The question is whether there is a subcollection $\mathcal{T}' \subseteq \mathcal{T}$ in which every element of U occurs exactly once. By taking the disjoint union of two isomorphic copies if necessary, we may assume $q \geq 2$.

Double-counting element–triple incidences gives

$$3|\mathcal{T}| = \sum_{T \in \mathcal{T}} |T| = \sum_{e \in U} |\{T \in \mathcal{T} : e \in T\}| = 3|U| = 9q.$$

Consequently, $|\mathcal{T}| = 3q$.

We construct an approval election as follows. For every triple $T \in \mathcal{T}$, introduce a candidate c_T and a private voter p_T. Thus

$$C := \{c_T : T \in \mathcal{T}\}, \qquad P := \{p_T : T \in \mathcal{T}\}, \qquad N := U \,\dot{\cup}\, P.$$

For each element voter $e \in U$, define $\mathcal{A}_e := \{c_T : T \in \mathcal{T},\ e \in T\}$, and for every private voter p_T, define $\mathcal{A}_{p_T} := \{c_T\}$. Finally, put $K := q$ and $x_{c_T} := \frac{1}{3}$ for every $T \in \mathcal{T}$. The construction is clearly polynomial. Since $|\mathcal{T}| = 3q$, it satisfies

$$|N| = |U| + |P| = 6q, \qquad |C| = 3q, \qquad |x| = \frac{|\mathcal{T}|}{3} = q = K.$$

We next verify all of the promised structural properties.

Regularity. Every element voter approves exactly three candidates, every private voter approves exactly one candidate, and every candidate is approved by its three element voters and its private voter. Thus every candidate has exactly four approving voters.

Let $\varnothing \neq S \subsetneq N$, and set $\Gamma(S) := \bigcup_{i \in S} \mathcal{A}_i$. Counting incidences between voters in S and candidates in $\Gamma(S)$, we obtain

$$|S| \leq \sum_{i \in S} |\mathcal{A}_i| = \sum_{c \in \Gamma(S)} |N(c) \cap S| \leq 4|\Gamma(S)|.$$

Therefore

$$|\Gamma(S)| \geq \frac{|S|}{4} > \frac{|S|}{6} = \frac{K|S|}{|N|}.$$

Hence the constructed election is regular.

Full column rank, antichainness, and positive duality. Order the rows first by the element voters U and then by the private voters P. If B is the U-by-$\mathcal{T}$ incidence matrix of the original RXC3 instance, then the complete approval matrix is

$$A = \begin{pmatrix} B \\ I_{3q} \end{pmatrix}.$$

The identity block implies $\operatorname{rank}(A) = 3q$, so A has full column rank.

Every column of A has exactly four ones. Moreover, the column corresponding to c_T contains the private row p_T, which occurs in no other column. The columns are therefore pairwise distinct. Since they all have the same cardinality, no one of their supports can properly contain another. Thus the column supports form an antichain.

Define $\alpha \in \mathbb{R}^N_{>0}$ by

$$\alpha_e := \frac{1}{6} \quad (e \in U), \qquad \alpha_{p_T} := \frac{1}{2} \quad (T \in \mathcal{T}).$$

For every candidate c_T,

$$\alpha(N(c_T)) = \sum_{e \in T} \alpha_e + \alpha_{p_T} = 3 \cdot \frac{1}{6} + \frac{1}{2} = 1.$$

Consequently, $A^\top \alpha = \mathbf{1}$, and hence $\mathcal{D}(A) \neq \varnothing$.

The supplied committee is a fractional Nash-core allocation. At x, every element voter has utility $u_e(x) = 3 \cdot \frac{1}{3} = 1$ for $e \in U$, whereas every private voter has utility $u_{p_T}(x) = \frac{1}{3}$ for $T \in \mathcal{T}$.

Let $\Phi(z) := \sum_{i \in N} \log u_i(z)$ be the unweighted Nash objective on the uncapped comparison domain $z \in \mathbb{R}^C_{\geq 0}$, $|z| \leq K$. If some coordinate z_{c_T} equals zero, then the private voter p_T has zero utility, and $\Phi(z) = -\infty$. Otherwise Φ is differentiable at z. At x, for each candidate c_T,

$$\begin{aligned} \frac{\partial \Phi}{\partial z_{c_T}}(x) &= \sum_{i : c_T \in \mathcal{A}_i} \frac{1}{u_i(x)} \\ &= \sum_{e \in T} \frac{1}{u_e(x)} + \frac{1}{u_{p_T}(x)} \\ &= 3 \cdot 1 + 3 \\ &= 6 = \frac{|N|}{K}. \end{aligned}$$

Since Φ is concave, every feasible z with finite objective satisfies

$$\begin{aligned} \Phi(z) &\leq \Phi(x) + \nabla \Phi(x)^\top (z - x) \\ &= \Phi(x) + 6\big(|z| - |x|\big) \\ &\leq \Phi(x). \end{aligned}$$

Thus x maximizes the Nash objective over the uncapped budget simplex. Taking $w_c = 1$ for all candidates makes the weight-selection coupling condition vacuous. Hence $(\mathbf{1}, x)$ is a fractional Nash-core pair.

For completeness, we also verify fractional-core membership directly. The vector α defined above satisfies $\alpha_i u_i(x) = 1/6 = K/|N|$ for every $i \in N$. Suppose that a coalition $S \neq \varnothing$ and a fractional committee $y \in [0,1]^C$ blocked x. Using $A^\top \alpha = \mathbf{1}$, nonnegativity, and strict improvement for all members of S, we would have

$$\begin{aligned}
|y| &= \sum_{c\in C} y_c \alpha(N(c)) \\
&= \sum_{i\in N} \alpha_i u_i(y) \\
&\geq \sum_{i\in S} \alpha_i u_i(y) \\
&> \sum_{i\in S} \alpha_i u_i(x) \\
&= \frac{|S|}{6} = \frac{K|S|}{|N|}.
\end{aligned}$$

This contradicts the blocking budget condition $|y| \leq K|S|/|N|$. Therefore x lies in the fractional core.

Residual floors. Since $0 < x_{c_T} < 1$ for every $T \in \mathcal{T}$, all candidates are fractional. Thus A is precisely the residual matrix, $f = x$, and $\kappa = \mathbf{1}^\top f = q$. For every element voter $e \in U$, $(Af)_e = 1$, while for every private voter p_T, $(Af)_{p_T} = \frac{1}{3}$. Hence the residual floor vector $h = \lfloor Af \rfloor$ is given by

$$h_e = 1 \quad (e \in U), \qquad h_{p_T} = 0 \quad (T \in \mathcal{T}).$$

Moreover, because $q \geq 2$ and $m = 3q$, we have $2 \leq \kappa = q \leq 3q - 2 = m - 2$.

It remains to establish the exact correspondence with RXC3. Suppose first that $\mathcal{T}' \subseteq \mathcal{T}$ is an exact cover of U. Then $|\mathcal{T}'| = q$, and the committee $W := \{c_T : T \in \mathcal{T}'\}$ has size $q = K$. Every element voter approves exactly one member of W, and every private voter has floor zero, so $u_i(W) \geq \lfloor u_i(x) \rfloor$ for every $i \in N$. Thus W is a feasible floor-preserving rounding.

Conversely, suppose that $W \subseteq C$ is a feasible floor-preserving rounding. Every element voter must receive utility at least one. Therefore

$$\begin{aligned}
3q &\leq \sum_{e\in U} u_e(W) \\
&= \sum_{c_T\in W} |T| \\
&= 3|W| \\
&\leq 3q.
\end{aligned}$$

Every inequality is consequently an equality. In particular, $|W| = q$, and $\sum_{e\in U} u_e(W) = 3q$. Since all $3q$ element voters have integer utility at least one, this equality forces $u_e(W) = 1$ for every $e \in U$. Thus the q triples corresponding to the candidates in W cover every element exactly once. They form an exact cover of U.

We have therefore constructed, in polynomial time, a Floor-Rounding yes-instance if and only if the original RXC3 instance is a yes-instance. This proves NP-hardness, and membership in NP completes the proof. □

Scope of Proposition 1.9. The proposition establishes hardness of the particular sufficient route used in the floor-rounding lemma: given a specified fractional core x, decide whether all utility floors of x can be preserved by an integral committee of budget at most K. It does not establish NP-hardness of discrete-core nonemptiness, nor does it imply that a no-instance of Floor-Rounding has an empty discrete core. Furthermore, although the reduction produces a full-column-rank positive-dual antichain satisfying the numerical residual-budget bounds of Proposition 1.8, it does not assert that the supplied fractional Nash-core pair minimizes the number of fractional coordinates among all fractional Nash-core pairs.

1.3.1 Residual floor regions

Fix a full-column-rank antichain $A \in \{0,1\}^{n\times m}$ and an integer residual budget κ. Define $\mathcal{U}(A,\kappa) := \{Az : z \in \{0,1\}^m,\ \mathbf{1}^\top z = \kappa\}$ to be the finite set of utility vectors obtainable by selecting exactly κ residual candidates. By the preceding definition, an integer vector $h \geq 0$ is implementable exactly when some $u \in \mathcal{U}(A,\kappa)$ satisfies $u \geq h$. A nonimplementable vector h is coordinatewise minimal if no smaller nonnegative integer vector is nonimplementable.

For an integer utility profile h, we define the fractional allocation region indexed by h and its closed relaxation to be

$$\mathcal{R}_<(A,\kappa,h) := \{f \in (0,1)^m : \mathbf{1}^\top f = \kappa,\ h \leq Af < h + \mathbf{1}\},$$
$$\mathcal{R}_\leq(A,\kappa,h) := \{f \in [0,1]^m : \mathbf{1}^\top f = \kappa,\ h \leq Af \leq h + \mathbf{1}\}.$$

All vector inequalities are componentwise.

2 The six-, seven-, and eight-voter cases

The three bounded-voter arguments use the same necessary blocker inequalities and one-deficit rounding principles. We collect these common ingredients first and then treat six, seven, and eight voters in separate subsections. The exact polyhedral facts used to verify the finite classifications are recorded in Appendix A.

2.1 Common blocker inequalities and rounding principles

2.1.1 Necessary blocker inequalities

Fix a residual integral committee $z \in \{0,1\}^m$ with $\mathbf{1}^\top z = \kappa$, and put $q := Az$ and $W := I \cup \operatorname{supp}(z)$. Thus, $q_i = (Az)_i$ is the number of selected candidates from F that voter i approves, and W is an integral committee.

Proposition 2.1 (Necessary conditions for a blocking coalition)**.** *If an integral committee T blocks W through a coalition S of size s, then*
(i) $\sum_{i\in S} \alpha_i(q_i + 1 - r_i) \leq 0$*;*
(ii) Every $i \in S$ satisfies $\lceil t/\alpha_i + q_i - r_i\rceil + 1 \leq \lfloor ts \rfloor$.

Proof. (i). Put $R := T \setminus I$ and $D := I \setminus T$. Strict integral improvement gives

$$u_i(R) - u_i(D) \geq q_i + 1 \quad \forall i \in S. \tag{7}$$

Every candidate in R lies in $F \cup O$, so Proposition 1.7(i) implies

$$\sum_{i\in S} \alpha_i u_i(R) \leq |R|. \tag{8}$$

The total normalized payment made by S to the fully selected candidates is $B_S := \sum_{i\in S} \alpha_i\beta_i = t|S| - \sum_{i\in S}\alpha_i r_i$. The retained full candidates $I \setminus D$ carry at most $|I \setminus D|$ of this payment. Since $w_c \leq 1$, ordinary α-mass dominates weighted payment mass, and therefore

$$\sum_{i\in S} \alpha_i u_i(D) \geq \max\{0, B_S - |I \setminus D|\}. \tag{9}$$

If $B_S \leq |I \setminus D|$, then $|R| \leq t|S| - B_S$ because $|I \setminus D| + |R| = |T| \leq t|S|$. If $B_S > |I \setminus D|$, equations (8) and (9) give

$$\sum_{i\in S} \alpha_i(u_i(R) - u_i(D)) \leq |R| + |I \setminus D| - B_S \leq t|S| - B_S.$$

Thus in both cases

$$\sum_{i\in S} \alpha_i(u_i(R) - u_i(D)) \le t|S| - B_S = \sum_{i\in S} \alpha_i r_i.$$

Comparison with (7) proves part (i).

(ii). By (6), the integer utility from the fully selected candidates satisfies $b_i \ge \lceil t/\alpha_i - r_i \rceil$. The incumbent committee gives voter i utility $b_i + q_i$. A strict integral improvement therefore gives at least $\lceil t/\alpha_i - r_i\rceil + q_i + 1 = \lceil t/\alpha_i + q_i - r_i\rceil + 1$ approved candidates. This cannot exceed the cardinality of the deviation, which is at most $\lfloor ts \rfloor$, proving part (ii). □

For a coalition S and a residual committee z, define the price margin $P_{S,z}(\alpha, f) := \sum_{j\in S} \alpha_j((Az)_j + 1 - (Af)_j)$. The price filter says that a blocking coalition must have nonpositive price margin.

2.1.2 One-deficit rounding principles

Definition 2.2 (One-deficit roundings and one-deficit rows)**.** Let h be nonimplementable. A *one-deficit rounding at voter i* is a vector $z^i \in \{0,1\}^m$ with $\mathbf{1}^\top z^i = \kappa$ and $Az^i \ge h - e_i$. Necessarily $h_i > 0$, $(Az^i)_i = h_i - 1$, and $(Az^i)_j \ge h_j$ for every $j \ne i$, for otherwise $Az^i \ge h$ would implement h. The set of one-deficit rows is $B(h) := \{i : h_i > 0 \text{ and } h - e_i \text{ is implementable}\}$.

The α-separated criterion.

Theorem 2.3 (α-separated one-deficit rounding theorem)**.** *Let z^i be a one-deficit rounding at voter i. If*

$$\alpha_i\delta_i < t(1 - \alpha_i), \tag{10}$$

$$\alpha_i\delta_i < \alpha_j(1 - \delta_j) \quad \forall j \in N \setminus \{i\}, \tag{11}$$

then $I \cup \operatorname{supp}(z^i)$ lies in the discrete core.

Proof. Let $q := Az^i$ and suppose a coalition S blocks the associated committee. If $i \notin S$, then $q_j + 1 - r_j \ge 1 - \delta_j > 0$ for every $j \in S$, so $P_{S,z^i}(\alpha, f) > 0$, contradicting Proposition 2.1(i). Hence $i \in S$.

At voter i the price coefficient is $q_i + 1 - r_i = -\delta_i$, while every other coefficient is at least $1 - \delta_j$. The price filter therefore implies $\sum_{j\in S\setminus\{i\}} \alpha_j(1 - \delta_j) \le \alpha_i\delta_i$. If $|S| \ge 2$, this contradicts (11).

It remains to rule out $S = \{i\}$. Condition (10) is equivalent to $t/\alpha_i - \delta_i > t$. For a one-deficit rounding, $q_i - r_i = -1 - \delta_i$, so $t/\alpha_i + q_i - r_i > t - 1$. Consequently, $\lceil t/\alpha_i + q_i - r_i\rceil + 1 > \lfloor t \rfloor$, contradicting Proposition 2.1(ii) with $s = 1$. □

Corollary 2.4 (Usable tight row)**.** *Suppose $i \in B(h)$ and $(Af)_i = h_i$. Then a core committee exists.*

Proof. Choose a one-deficit rounding z^i. Here $\delta_i = 0$. Since $h_i > 0$, voter i is incident with a fractional column. If the only fractional support containing i were the singleton $\{i\}$, then distinctness and antichainness would force every other fractional support to omit i, and r_i would equal one fractional coordinate in $(0, 1)$, contradicting the positive integer equality $r_i = h_i$. Thus some fractional column contains i and another voter. Its total α-mass is one, and all coordinates of α are positive, so $\alpha_i < 1$.

Condition (10) holds because $\delta_i = 0$ and $\alpha_i < 1$. Condition (11) holds because $\alpha_j > 0$ and $\delta_j < 1$ for every $j \ne i$. Therefore Theorem 2.3 applies, independently of the number of voters. □

2.2 Six voters

The six-row calculation deliberately examines a larger universe than the analytic reduction requires: it includes every full-column-rank six-row antichain and does not first impose positive duality.

Computer Lemma 2.5 (Six-row tight-row classification)**.** *Let $A \in \{0,1\}^{6\times m}$ be a full-column-rank antichain, where $4 \le m \le 6$, and let $2 \le \kappa \le m-2$. Let h be nonimplementable and suppose $\mathcal{R}_<(A,\kappa,h) \neq \varnothing$. Then, for every $f \in \mathcal{R}_<(A,\kappa,h)$, there exists $i \in B(h)$ such that $(Af)_i = h_i$.*

Exact computer-assisted proof. The script `scripts/regenerate_n6.sh` orchestrates the complete six-row run. Canonical enumeration is performed by `src/common/enumerate_antichains.cpp`. It generates every full-column-rank six-row antichain for $m = 4, 5, 6$. The canonical output `data/n6/antichains.txt` contains respectively 279, 712, and 1,162 matrices.

For each matrix and residual budget, the program `src/common/generate_minimal_holes.cpp` enumerates every integral utility vector Az and generates every coordinatewise-minimal nonimplementable floor. The programs `src/common/filter_holes_subset.cpp` and `src/common/filter_holes_coverdual.cpp` apply the exact subset and cover-dual exclusions. The upper cones of the surviving minimal floors are generated by `src/common/generate_upper_cells.cpp` and filtered again by `src/common/filter_cells_coverdual.cpp`. Appendix A proves that these stages are complete and that every discarded floor region is excluded by an exact certificate.

It remains to exclude a counterexample inside each retained floor region. The integral utility list determines $B(h)$ exactly. A counterexample would be a point $f \in \mathcal{R}_<(A,\kappa,h)$ satisfying $(Af)_i > h_i$ for every $i \in B(h)$. For a nonminimal floor this is equivalent to a positive optimum in the exact rational program

$$
\begin{aligned}
\max \quad & \varepsilon, & \\
\text{subject to} \quad & \varepsilon \le f_c \le 1-\varepsilon & \text{for every } c \in [m], \\
& \mathbf{1}^\top f = \kappa, & \\
& h \le Af \le h + \mathbf{1} - \varepsilon \mathbf{1}, & \\
& (Af)_i \ge h_i + \varepsilon & \text{for every } i \in B(h), \\
& \varepsilon \ge 0. &
\end{aligned}
$$

For a coordinatewise-minimal floor, a possible counterexample has either strict surplus on all six rows or a boundary equality $(Af)_q = h_q = 0$ at some $q \notin B(h)$ together with strict surplus on every row in $B(h)$. These are the exhaustive branches implemented by `src/common/classify_cells_exact.cpp` and the independent arbitrary-precision implementation `src/common/classify_cells_bigint.cpp`.

The proof-relevant final input is `data/n6/final_cells.txt`, containing 46 records. The two classifiers produce byte-identical empty counterexample outputs, bound in the release as `data/n6/bad_cells.empty` and `data/n6/surplus_cells.empty`. The fast replay `scripts/verify_n6.sh` terminates with

```
PASS n=6 antichains=2153 cells=46 bad=0 surplus=0.
```

Consequently no retained floor region contains a counterexample, which proves the lemma. □

2.3 Seven voters

At seven rows, a usable tight row is no longer present throughout every residual floor region. The remaining regions admit an α-separated one-deficit rounding.

Table 1: Exact six-row classification.

Object	Count
Full-rank antichains with $m = 4$	279
Full-rank antichains with $m = 5$	712
Full-rank antichains with $m = 6$	1,162
Full-rank antichains, total	2,153
Raw coordinatewise-minimal targets	32,286
Targets surviving all subset inequalities	6,053
Targets surviving the exact cover-dual filter	46
Upper-cone floor vectors generated	597
Floor vectors entering exact classification	46
Regions containing a classified counterexample	0

Computer Lemma 2.6 (Seven-row rounding classification)**.** *Let $A \in \{0,1\}^{7\times m}$ be a full-column-rank antichain with $\mathcal{D}(A) \neq \varnothing$, where $4 \le m \le 7$ and $2 \le \kappa \le m-2$. Let h be nonimplementable and suppose $\mathcal{R}_<(A,\kappa,h) \neq \varnothing$. For every $\alpha \in \mathcal{D}(A)$ and $f \in \mathcal{R}_<(A,\kappa,h)$, at least one of the following holds:*

(a) *there exists $i \in B(h)$ with $(Af)_i = h_i$;*

(b) *there exists $i \in B(h)$ such that*

$$\alpha_i\delta_i < \frac{\kappa}{7}(1-\alpha_i),$$
$$\alpha_i\delta_i < \alpha_j(1-\delta_j) \qquad \textit{for every } j \neq i.$$

Exact computer-assisted proof. The complete run is orchestrated by `scripts/regenerate_n7.sh`. The programs `src/common/enumerate_antichains.cpp`, `src/n7/filter_positive_alpha.cpp`, and `src/n7/enumerate_positive_dual.cpp` generate the complete collection of positive-dual matrices. The shared minimal-nonimplementable-floor, subset-filter, cover-filter, and open-region programs from the six-row proof reduce this collection to the 30,929 records in `data/n7/final_cells.txt`.

The exact region classifiers find no counterexample to the tight-row branch and no nonprimitive all-surplus region. They identify exactly 298 primitive all-surplus regions. Their canonical list is `data/n7/surplus_cells.txt`. The file `data/n7/puncture_certificates.json` stores an explicit one-deficit rounding for every eligible voter–region assertion used by the separation proof; `src/n7/verify_puncture_certificates.py` directly recomputes every such integer utility inequality.

Zero rows require no separation certificate. If row q of A is zero, then feasibility forces $h_q = \delta_q = 0$. Deleting that row leaves a full-column-rank six-row antichain, forces $m \le 6$, and preserves κ, the open region in the f variables, and $B(h)$ on the remaining rows. Computer Lemma 2.5 therefore supplies a usable tight row, so alternative (a) holds. Consistently, none of the 298 all-surplus records has a zero row; in every one of them $B(h) = [7]$.

Of the 298 regions, 294 arise from square 7×7 matrices. The arbitrary-precision program `src/n7/verify_square_surplus.cpp` verifies the separation inequalities over the complete closed floor polytope at the unique positive dual. The remaining four arise from 7×6 matrices. The program `src/n7/verify_rectangular_surplus.cpp` treats the choice of deficient voter as part of the certificate. Because these matrices have no zero row, the closure of $\mathcal{D}(A)$ is a compact line segment: full column rank gives a one-dimensional affine solution set, and every coordinate is bounded above by one. Each endpoint has a zero coordinate. A fixed index cannot work over the whole segment: if $\alpha_j = 0$, $i \neq j$, and $\delta_i > 0$, then the pairwise margin against j equals $-\alpha_i\delta_i < 0$. The verifier therefore partitions the segment into subintervals on which a specified coordinate α_i is minimal and may use a different eligible index on different subintervals. On each subinterval it checks the singleton bound exactly and uses $\alpha_i \le \alpha_j$ together with $\delta_i + \delta_j < 1$ for the pairwise bounds. If the maximum of $\delta_i + \delta_j$ on the closed floor polytope is exactly one, an additional common-slack linear program verifies that equality cannot occur in the open region. Proposition A.1(i)–(ii) then gives strict separation throughout the positive-dual interval and open floor region. Thus the programs verify alternative (b) over the whole real domain, not merely at sampled points.

The independent signed-rational and arbitrary-precision classifiers produce the same 298-record file and an empty counterexample output. The fast replay `scripts/verify_n7.sh` terminates with

```
PASS n=7 antichains=61846 cells=30929 surplus=298 bad=0.
```

Appendix A proves completeness of the enumeration and soundness of every exclusion. Therefore every admissible seven-row floor region satisfies alternative (a) or (b), proving the lemma. □

Table 2: Exact seven-row classification.

Object	Count
Positive-dual antichains, $m = 4, 5, 6, 7$	61,846
Matrix–budget cases	208,472
Raw coordinatewise-minimal targets	3,200,557
Targets surviving all subset inequalities	888,719
Minimal targets surviving exact cover filtering	16,854
Upper-cone floor vectors generated	1,579,161
Floor vectors entering exact classification	30,929
Exact rational region linear programs	43,353
Primitive all-surplus regions	298
Other unresolved regions	0

Theorem 2.7 (The cases of at most seven voters)**.** *Every approval-based committee election with at most seven voters has a committee in the discrete core.*

Proof. We use strong induction on the number of voters. The cases $n \le 5$ are known from [3]. For $n \in \{6, 7\}$, an irregular election reduces by Lemma 1.5 to an election with fewer voters. In a regular election, delete candidates approved by no voter. If at most K candidates remain, select all of them; otherwise Theorem 1.6 applies, and we choose a minimum-fractional Nash-core pair. If the residual floor is implementable, Lemma 1.3 applies. Otherwise Proposition 1.8 gives an admissible residual matrix.

For $n = 6$, Computer Lemma 2.5 gives a usable tight row, and Corollary 2.4 gives a core committee. For $n = 7$, apply Computer Lemma 2.6. Alternative (a) again gives a usable tight row. In alternative (b), the entitlement satisfies $t = K/7 \ge \kappa/7$, so the two displayed inequalities in the computer lemma imply (10) and (11). Theorem 2.3 then gives a core committee. □

2.4 Eight voters

The seven-row separation inequalities do not certify every eight-row residual floor region. The eight-voter argument first derives the individual saturation lower bounds encoded by $\lambda_{\kappa,s,d}$; the exact classification then tests coalition price margins on the restricted price domain.

2.4.1 Saturation inequalities for eight voters

We use Proposition 2.1(ii) to derive a necessary condition that depends only on the residual fractional part containing A, κ, and h, but not the original budget K or the fully selected candidates I.

For integers $\kappa \geq 1$, $1 \leq s \leq 8$, and $d \in \mathbb{Z}$, define

$$\lambda_{\kappa,s,d} := \min\left(\left\{\frac{1}{s}\right\} \cup \left\{\frac{B}{8(\lfloor Bs/8 \rfloor + d)} : B = \kappa, \dots, \kappa+7,\ \lfloor Bs/8 \rfloor + d > 0\right\}\right). \tag{12}$$

The set in (12) is nonempty because it contains $1/s$.

Lemma 2.8. *Consider an eight-voter election. If a coalition S of size s blocks a residual committee z, then every $i \in S$ satisfies $\alpha_i > \lambda_{\kappa,s,h_i-(Az)_i}$.*

Proof. Put $d := h_i - (Az)_i$ and $L := \lfloor Ks/8 \rfloor$. Since $r_i = h_i + \delta_i$ with $\delta_i < 1$, Proposition 2.1(ii) gives $K/(8\alpha_i) - d - \delta_i \leq L - 1$, and hence $K/(8\alpha_i) < L + d$. Thus $L + d > 0$ and $\alpha_i > K/[8(\lfloor Ks/8 \rfloor + d)]$. It remains to minimize the right-hand side over all integers $K \geq \kappa$.

Fix a residue class modulo eight and let $B_r \in \{\kappa, \dots, \kappa+7\}$ be its first representative at least κ. Write $K = B_r + 8a$ and $D_r := \lfloor B_r s/8 \rfloor + d$. On this residue class the ratio is $g_r(a) = (B_r + 8a)/[8(D_r + as)]$. Its successive differences have constant sign, the sign of $8D_r - sB_r = 8d - (B_r s \bmod 8)$, and $g_r(a) \to 1/s$ as $a \to \infty$. If $D_r > 0$, the infimum on the class is either $g_r(0)$ or $1/s$. If $D_r \leq 0$, then $d \leq 0$. The displayed sign is strictly negative: this is immediate when $d < 0$, while for $d = 0$ the condition $D_r = 0$ forces $0 < B_r s < 8$, so $(B_r s \bmod 8) > 0$. Hence the admissible tail is decreasing and its infimum is $1/s$. Taking the minimum over the eight residue classes gives (12) and proves the lemma. ☐

For a one-deficit rounding z^i, define its singleton margin

$$G_i(\alpha, f) := \frac{\kappa}{8}(1 - \alpha_i) - \alpha_i((Af)_i - h_i) = \frac{\kappa}{8}(1 - \alpha_i) - \alpha_i \delta_i.$$

Theorem 2.9. *Consider an eight-voter election.*

(i) Let z^i be a one-deficit rounding at voter i. Suppose $G_i(\alpha, f) > 0$ and, for every coalition $S \ni i$ with $|S| \geq 2$, the inequalities $\alpha_j > \lambda_{\kappa,|S|,h_j-(Az^i)_j}$ for every $j \in S$ imply $P_{S,z^i}(\alpha, f) > 0$. Then $I \cup \operatorname{supp}(z^i)$ lies in the discrete core.

(ii) Let $E \subseteq [8]$ be nonempty. Suppose that for every $i \in E$ there is a one-deficit rounding z^i satisfying the coalition-margin condition in part (i), and suppose

$$\sum_{i \in E} G_i(\alpha, f) > 0. \tag{13}$$

Then at least one of the committees $I \cup \operatorname{supp}(z^i)$, $i \in E$, lies in the discrete core.

Proof. For part (i), note first that $(Az^i)_j + 1 - r_j \geq 1 - \delta_j > 0$ for every $j \neq i$, whereas the coefficient at i is $-\delta_i \leq 0$. Hence any coalition satisfying the necessary price inequality must contain i.

A singleton block is impossible. Indeed, $G_i > 0$ implies $\alpha_i < 1$ and $\alpha_i \delta_i < (\kappa/8)(1 - \alpha_i) \leq t(1 - \alpha_i)$, so the singleton argument in Theorem 2.3 applies. If a blocking coalition has size at least two, Lemma 2.8 places its price vector in the stipulated strict domain, and the resulting positive price margin contradicts Proposition 2.1(i).

For part (ii), equation (13) gives $G_i(\alpha, f) > 0$ for some $i \in E$. Part (i) applies to that one-deficit rounding. The successful index may depend on the actual pair (α, f); the conclusion is existential. ☐

2.4.2 Exact eight-row classification

Definition 2.10 (Fixed and adaptive certificates)**.** A fixed certificate stores one index $i \in B(h)$, one one-deficit rounding z^i, and exact proofs that $G_i(\alpha, f) > 0$ and every required coalition price margin is positive throughout the relevant floor and dual domains. An adaptive certificate stores a nonempty set $E \subseteq B(h)$, a one-deficit rounding z^i for every $i \in E$, exact coalition certificates for every rounding, and an exact proof that $\sum_{i \in E} G_i(\alpha, f) > 0$ throughout the domain.

Computer Lemma 2.11 (Eight-row rounding classification)**.** *Let $4 \le m \le 8$ and $2 \le \kappa \le m-2$. Let $A \in \{0,1\}^{8\times m}$ be a full-column-rank antichain with $\mathcal{D}(A) \neq \varnothing$. Let h be nonimplementable and suppose $\mathcal{R}_<(A, \kappa, h) \neq \varnothing$. At least one of the following holds:*

(I) *for every $f \in \mathcal{R}_<(A, \kappa, h)$, there exists $i \in B(h)$ such that $(Af)_i = h_i$;*

(II) *there exist $i \in B(h)$ and a one-deficit rounding z^i such that, for every $f \in \mathcal{R}_<(A, \kappa, h)$ and $\alpha \in \mathcal{D}(A)$, $G_i(\alpha, f) > 0$, and for every coalition $S \ni i$ with $|S| \ge 2$,*

$$\Big[\alpha_j > \lambda_{\kappa,|S|,h_j-(Az^i)_j} \text{ for every } j \in S\Big] \Longrightarrow P_{S,z^i}(\alpha, f) > 0;$$

(III) *there exist a nonempty set $E \subseteq B(h)$ and one-deficit roundings z^i, $i \in E$, such that the coalition implication in* (II) *holds for every $i \in E$ and $\sum_{i \in E} G_i(\alpha, f) > 0$ for every $f \in \mathcal{R}_<(A, \kappa, h)$ and $\alpha \in \mathcal{D}(A)$.*

Exact computer-assisted proof. The canonical positive-dual matrix hierarchy is generated by `src/n8/m8_canonical_enumerator.cpp`. The square-stage floor-region scanner is `src/n8/m8_floor_cell_scanner.cpp`. The lower-column scanner is `src/n8/eight_row_floor_cell_scanner_template.cpp`. A floor region is recorded as unresolved exactly when the tight-row classifier finds a point with strict surplus on every row in $B(h)$; hence the unresolved regions are precisely those not already covered by alternative (I). If $B(h) = \varnothing$, the strict-surplus condition is vacuous, so the scanner records the region as unresolved and increments its explicit `B_empty` counter rather than treating alternative (I) as proved. Such a record has no permissible index for alternative (II) and no nonempty set for alternative (III), and therefore would produce a failure record. Every completed scan reports `B_empty=0`; hence all unresolved regions have $B(h) \neq \varnothing$ before certificate replay.

The program `src/n8/m8_certificate_proposer.cpp` may use floating point to choose promising fixed or adaptive one-deficit roundings. Its verdict is not trusted. The stored certificate is accepted only after exact replay by `src/n8/m8_exact_checker_ll.cpp` and the independent GMP implementation `src/n8/m8_exact_checker_gmp.cpp`. These programs recompute the one-deficit inequalities, the finite saturation threshold, the emptiness of every restricted price domain, the singleton or adaptive-sum margin, and every coalition price margin.

The eight-row $m = 6$ records are checked by `src/n8/m6_exact_certificate_checker.cpp`; the $m = 7$ records are checked by `src/n8/exact_cps_fullsat_checker.cpp`. For $m = 8$, the program `src/n8/m8_record_checker.cpp` proves that the records for unresolved regions are exactly the disjoint union of accepted certificates and failure records. The program `src/n8/m8_kernel_list_verifier.cpp` independently verifies sortedness, uniqueness, antichainness, full rank, and positive duality of every stored square matrix.

For $m = 4, 5$, the exact scan finds no unresolved regions. For $m = 6, 7, 8$, every unresolved region has a fixed or adaptive certificate and the failure files contain zero records. The exact totals are shown in Table 3. The complete replay is exposed by `scripts/verify_n8.sh` and by the package-wide `scripts/quick_verify.sh`. In the released artifact these commands terminate successfully after the lower-column checks, square-matrix audit, record-set comparison, and both square-stage exact replays. The exact vertex, endpoint, and recession reductions are justified by Proposition A.1. Appendix A proves that these checks establish the stated universal inequalities on the complete floor and dual domains. Therefore the three alternatives are exhaustive, proving the lemma. □

Table 3: Exact eight-row residual census, indexed by the number m of fractional columns.

m	positive-dual matrices	matrix–budget cases	feasible regions	unresolved regions	fixed certificates	adaptive certificates
4	4,779	4,779	22	0	0	0
5	56,479	112,958	84	0	0	0
6	561,445	1,684,335	5,766	168	163	5
7	3,541,727	14,166,908	89,286	36,128	36,119	9
8	9,105,190	45,525,950	1,081,420	1,049,187	1,049,177	10
Total	13,269,620	61,494,930	1,176,578	1,085,483	1,085,459	24

For the square $m = 8$ stage, the budget-by-budget counts are

κ	2	3	4	5	6
feasible regions	59,523	262,243	437,888	262,243	59,523
unresolved regions	59,358	258,574	413,323	258,574	59,358
fixed	59,350	258,572	413,323	258,574	59,358
adaptive	8	2	0	0	0

The complement symmetry between $\kappa = 2$ and 6, and between $\kappa = 3$ and 5, is an additional exact consistency check.

Completion of the main theorem

Proof of Theorem 1.1. The result for at most seven voters is Theorem 2.7. Consider an election with eight voters. If it is irregular, Lemma 1.5 reduces it to an election with fewer voters. Suppose it is regular. Delete candidates approved by no voter. If at most K candidates remain, select all of them; otherwise Theorem 1.6 applies, and we choose a minimum-fractional Nash-core pair.

If the residual floor h is implementable, Lemma 1.3 gives a core committee. Otherwise Proposition 1.8 gives a positive-dual full-column-rank antichain with $4 \le m \le 8$ and $2 \le \kappa \le m - 2$. Apply Computer Lemma 2.11 at the actual pair (f, α). Branch (I) is handled by Corollary 2.4; branch (II) by Theorem 2.9(i); and branch (III) by Theorem 2.9(ii). The branches are exhaustive. □

Discussion

The six-, seven-, and eight-voter cases share the same fractional foundation and structural reduction, but the integral rounding method becomes progressively stronger. Six rows are resolved by a tight one-deficit row. Seven rows require a one-deficit rounding whose fractional debt is separated from every other voter's one-unit gap. At eight rows, that separation criterion no longer covers all residual regions. The individual saturation inequality supplies the missing information: a voter with a given floor deficit can belong to a blocking coalition only if her Nash price exceeds an explicit threshold. Restricting the price domain by these thresholds makes the coalition price filter decisive.

The theorem is existential. The proof does not by itself yield a polynomial-time algorithm for arbitrary input elections, because computing a minimum-fractional Nash-core pair and locating the corresponding canonical residual certificate are separate tasks. It also remains a bounded-voter theorem, not a solution of the unrestricted discrete-core problem. Nevertheless, the price–saturation threshold is independent of the omitted fully selected candidates and suggests a sharper starting point for the next bounded-voter frontier.

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A Computational verification and supporting artifact

This appendix records the polyhedral facts used by the exact checkers and proves that the programs named in the three computer-lemma proofs establish the required finite statements. The detailed compiler commands, record formats, and run logs are part of the artifact; the mathematical obligations and their source-level realization are recorded here.

A.1 Mathematical and computational verification

A.1.1 Polyhedral facts used by the exact checkers

Proposition A.1 (Polyhedral reductions used by the exact checkers)**.** *Assume $\mathcal{R}_<(A,\kappa,h) \neq \varnothing$.*

(i) The fractional allocation region $\mathcal{R}_<(A,\kappa,h)$ is dense in its closed relaxation $\mathcal{R}_\le(A,\kappa,h)$: $\mathrm{cl}(\mathcal{R}_<(A,\kappa,h)) = \mathcal{R}_\le(A,\kappa,h)$. Consequently, every continuous affine function L satisfies $\inf_{f\in\mathcal{R}_<(A,\kappa,h)} L(f) = \min_{f\in\mathcal{R}_\le(A,\kappa,h)} L(f)$.

(ii) If $Q \subseteq \mathbb{R}^m$ is a nonempty compact convex set and $L(\alpha,f)$ is affine in α for every fixed $f \in Q$, then $g(\alpha) := \min_{f\in Q} L(\alpha,f)$ is concave. In particular, if g is concave on $[a,b]$, is nonnegative at both endpoints, and is positive at at least one endpoint, then $g(x) > 0$ for every $x \in (a,b)$.

(iii) If $A \ge 0$, $d \ge 0$, and $A^\top d = 0$, then d is supported only on zero rows of A. Along such recession directions, one-deficit singleton margins and adaptive sums are constant, while coalition price margins are nondecreasing.

(iv) If P and Q are nonempty bounded polytopes and $B(x,y)$ is affine in each variable separately, then the minimum of B on $P \times Q$ is attained at a pair of vertices.

Proof. For part (i), the inclusion from left to right is immediate. Fix $f^\circ \in \mathcal{R}_<(A,\kappa,h)$. For every $\bar f \in \mathcal{R}_\le(A,\kappa,h)$ and $\varepsilon \in (0,1]$, the point $f_\varepsilon := (1-\varepsilon)\bar f + \varepsilon f^\circ$ lies in the open region and converges to $\bar f$ as $\varepsilon \downarrow 0$. Continuity and compactness give the assertion about affine functions.

For part (ii), if $0 \le \theta \le 1$, then

$$\begin{aligned} g(\theta\alpha + (1-\theta)\alpha') &= \min_{f\in Q}(\theta L(\alpha,f) + (1-\theta)L(\alpha',f)) \\ &\ge \theta g(\alpha) + (1-\theta)g(\alpha'). \end{aligned}$$

For $x = \theta a + (1-\theta)b$ with $0 < \theta < 1$, the same inequality and the endpoint assumptions give $g(x) > 0$.

For part (iii), every column c satisfies $0 = (A^\top d)_c = \sum_i A_{ic} d_i$. All summands are nonnegative, so $d_i > 0$ implies that row i is zero. The deficient voter in a one-deficit rounding has a nonzero row, hence its singleton coefficient is unchanged along the ray. On a zero row one has $(Az)_i = (Af)_i = 0$, so a coalition price margin has recession slope 1 when that row belongs to the coalition and 0 otherwise.

For part (iv), choose a minimizing pair (x^*, y^*). With y^* fixed, the affine function $B(\,\cdot\,, y^*)$ has a minimizing vertex v of P; with v fixed, $B(v, \cdot\,)$ has a minimizing vertex w of Q. Then (v,w) is minimizing. □

Part (i) makes a strictly positive minimum on the closed floor polytope sufficient for strict positivity on the open region, but not necessary: a zero minimum may occur only at the excluded relative boundary. The exact checkers use the following conventions. When no separate relative-boundary argument is implemented, a record is accepted only if the exact closed-domain minimum is strictly positive; this applies to the eight-row $m = 6$ vertex-pair checks and to the $m = 8$ singleton, adaptive-sum, and exact price-LP routes. Other routes exploit the open inequalities explicitly. The seven-row rectangular verifier uses an auxiliary common-slack program when a closed-floor bound $\delta_i + \delta_j \le 1$ is tight, while the seven-row square verifier searches directly for an open-region conjunction of failure inequalities using a common slack variable. In the square $m = 8$ coalition checker, a nonnegative floor-level reserve is sufficient because

$$P_{S,z}(\alpha, f) = \sum_{j \in S} \alpha_j \big((Az)_j - h_j\big) + \sum_{j \in S} \alpha_j (1 - \delta_j) > 0$$

whenever the first sum is nonnegative. The strict inequality holds throughout $\mathcal{R}_<(A, \kappa, h)$, and the execution summary records this route in the `open_floor_checks` counter. On a dual interval, part (ii) permits a zero endpoint minimum only when the other endpoint minimum is positive. Thus a zero closed-domain minimum is accepted only through one of these explicit relative-interior arguments; otherwise the checker rejects it. The reported strictly positive minima concern the routes that require positive closed-domain values and do not include the open-floor reserve route.

For the eight-voter price checks, fix a coalition S and a one-deficit rounding z. To test whether S can block the committee produced by z, consider the price vectors $\alpha \in \mathbb{R}^8_{\geq 0}$ satisfying $A^\top \alpha = \mathbf{1}$ and the weak lower bounds $\alpha_j \geq \lambda_{\kappa, |S|, h_j - (Az)_j}$ for every $j \in S$. If S actually blocks, its Nash price vector has positive coordinates and satisfies all these lower bounds strictly. Thus every defining inequality has positive slack at this vector, so a sufficiently small neighborhood within the affine space $A^\top \alpha = \mathbf{1}$ remains feasible. The vector therefore lies in the relative interior of the set, which must consequently be nonempty.

For $m = 8$, the equation $A^\top \alpha = \mathbf{1}$ has a unique rational solution, so this set contains at most one point. For $m = 7$, the set is an interval, ray, point, or empty, and Proposition A.1(ii)–(iii) reduces universal strict positivity to exact endpoint and recession checks. For $m = 6$, it lies in a two-dimensional affine space; Proposition A.1(iv) reduces each bounded product-domain check to exact vertex pairs, while part (iii) handles unbounded rays. The exact checker accepts a boundary-zero ray certificate only when the zero occurs at an excluded endpoint and the recession slope is strictly positive; a margin that is identically zero is rejected. For $m = 4, 5$, the eight-row computation finds no unresolved regions.

A.1.2 Completeness and soundness of the finite verification

Proposition A.2 (Soundness and completeness of the finite verification)**.** *The supplied pipeline has the following properties:*

(i) *up to voter and candidate relabeling, every admissible residual matrix is enumerated;*
(ii) *every nonimplementable floor with a nonempty open region is examined unless it is removed by an exact infeasibility certificate;*
(iii) *every subset or cover-dual rejection is sound;*
(iv) *every retained six-, seven-, or eight-row record satisfies the relevant computer-lemma branch throughout its complete real floor and dual domains;*
(v) *every key for an unresolved eight-row region occurs exactly once in the accepted certificate file or the failure file, and every failure file has zero records.*

Proof. Matrix-enumeration completeness and the enumerator code. A candidate support is encoded as an n-bit mask. The shared program `src/common/enumerate_antichains.cpp` uses canonical augmentation under simultaneous voter relabeling and candidate permutation for the six- and seven-row base levels. Every antichain with $m+1$ columns contains an antichain with m columns; by induction, an isomorphic copy of such a parent is generated, and adjoining the deleted column generates a child in the desired orbit. Exact canonicalization removes duplicate children. The seven-row square enumerator is `src/n7/enumerate_positive_dual.cpp`; the eight-row hierarchy is `src/n8/m8_canonical_enumerator.cpp`. Deleting a column from a full-column-rank positive-dual antichain preserves antichainness, full column rank, and the same positive solution of the remaining equations. Therefore extending only positive-dual parents is complete. The final square list is audited independently by `src/n8/m8_kernel_list_verifier.cpp`.

Floor completeness and the generation code. For fixed (A, κ), implementable floors form the down-closure of the finite set $\mathcal{U}(A, \kappa)$. The program `src/common/generate_minimal_holes.cpp` starts from the zero vector and processes the vectors in $\mathcal{U}(A, \kappa)$. Whenever a current vector g is dominated by a new utility u, it replaces g by the vectors obtained by raising one coordinate to $u_i + 1$ and retains only coordinatewise-minimal vectors. After processing any initial segment of the utility list, the maintained set is exactly the set of minimal vectors not dominated by that segment. This invariant proves that all minimal nonimplementable floors are generated. The program `src/common/generate_upper_cells.cpp` enumerates their entire upper cones in the finite degree-bounded box and removes duplicate records exactly. The eight-row scanner implements the same invariant in `src/n8/m8_floor_cell_scanner.cpp` and `src/n8/eight_row_floor_cell_scanner_template.cpp`.

Filter soundness and the filter code. The subset program `src/common/filter_holes_subset.cpp` checks, for every voter subset S,

$$\sum_{i \in S} h_i \leq \max_{0 \leq f \leq 1,\ \mathbf{1}^\top f = \kappa} \sum_{i \in S} (Af)_i.$$

The exact maximum is the sum of the κ largest column intersections with S, so a violation proves that the floor region is empty. The cover programs `src/common/filter_holes_coverdual.cpp` and `src/common/filter_cells_coverdual.cpp` use floating point only to propose a nonnegative vector λ. They then recompute exactly the dual lower bound

$$\lambda^\top h - \sum_{c=1}^{m} \big((A^\top \lambda)_c - 1\big)_+.$$

A floor is rejected only when this exact value is strictly larger than κ. Weak duality then proves that its open floor region is empty.

Local soundness and the checker code. The six- and seven-row classifiers solve the exact common-slack linear programs using two arithmetic implementations: `src/common/classify_cells_exact.cpp` uses normalized signed rationals with overflow checks, while `src/common/classify_cells_bigint.cpp` uses Boost arbitrary-precision integers and rationals. The seven-row square and rectangular verifiers reduce universal real-domain claims to exact rational linear-program optima, interval endpoints, and recession slopes.

For eight rows, an accepted record stores the canonical matrix and floor key, the residual budget, the one-deficit rounding or roundings, and the data needed to replay every universal margin claim. The exact checkers recompute committee cardinality, one-deficit inequalities, positive-dual equations, all saturation thresholds, emptiness of discarded saturation domains, singleton or adaptive sum margins, and every coalition price margin. For $m = 8$, the price vector is unique. For $m = 7$, exact endpoint and recession checks suffice. For $m = 6$, the checker enumerates the vertices of the two-dimensional price domain and the vertices of the closed floor polytope; a bilinear margin attains its minimum at a vertex pair. Hence acceptance proves the certificate throughout the full domain, not merely at sampled points.

Coverage and the record-checker code. The program `src/n8/m8_record_checker.cpp` sorts and compares canonical record keys and verifies

$$\{\text{unresolved regions}\} = \{\text{accepted certificates}\} \,\dot{\cup}\, \{\text{failure records}\}.$$

The lower-column checkers perform the corresponding coverage checks for the $m = 6$ and $m = 7$ records. The released failure files contain zero records. This proves all five assertions. $\square$

A.1.3 Exact execution summary

For the eight-column stage, the record checker reports

```
PASS hard_records=1049187 certificate_records=1049187 failures=0.
```

The signed-rational and GMP replays agree on

```
PASS certs=1049187 fixed=1049177 adaptive=10
saturation_skips=122280434 open_floor_checks=6543104
exact_price_LPs=4430958 min_singleton_or_sum=1/1008
min_exact_price=1/11.
```

The two route-specific counters partition the same 10,974,062 non-saturation coalition checks. Their split can change when the proposal-only search selects a different exact-valid committee stream; it is not an additional proof invariant. The deposited stream and its exact replay are documented in `docs/CERTIFICATE_STREAM_PORTABILITY.md`. The square-matrix audit reports

```
PASS kernels=9105190 sorted_unique=1 antichain=1 full_rank=1
positive_dual=1 max_abs_det=56 max_abs_cramer=32 min_abs_det=1.
```

The eight-row $m = 6$ replay verifies 163 fixed and five adaptive certificates, with minimum singleton/adaptive margin 1/48 and minimum price margin 1/12. The $m = 7$ replay verifies 36,119 fixed and nine adaptive certificates with no failed coalition check. Together with Proposition A.2, these exact executions prove Computer Lemmas 2.5, 2.6, and 2.11.

A.2 Artifact, proof-to-code correspondence, and reproduction

The proof artifact accompanying this note is distributed separately through a versioned release. It contains the following proof-relevant components.

1. Canonical enumerators and exact matrix-list verifiers for the six-, seven-, and eight-row computations.
2. Exact generators of minimal nonimplementable vectors, subset filters, cover-dual replay code, and strict floor-region scanners.
3. Six-row tight-row classification code and its proof-relevant inputs and outputs.
4. Seven-row floor-region classification and surplus-separation verifiers, including the four rectangular certificates.
5. Eight-row $m = 4, 5, 6, 7, 8$ scans for unresolved regions, certificate generators, exact replay checkers, and record-set checkers.
6. The complete unresolved-region, certificate, and zero-failure record files, including the four principal eight-column binary files.
7. An arbitrary-precision exact replay for every final certificate family; the eight-column package includes both the signed-rational and GMP replays described above.
8. A machine-readable results summary, a complete SHA–256 manifest, build scripts, a quick verification script, and a full from-scratch regeneration script.
9. A platform-independent specification of every binary record format, including endianness, fixed-width field types, record size, file magic, and format version.
10. A clean-machine reproduction report recording operating system, compiler, exact-library versions, CPU, thread count, peak memory, disk use, and run time.

The proof-complete release compiles the canonical enumerators, exact floor-region scanners, certificate proposers, signed-rational checkers, GMP checker, record checkers, and matrix-list verifiers. It then verifies or regenerates the positive-dual hierarchies, scans every residual budget, checks record coverage, and replays every certificate exactly. The data-complete package contains all stored inputs, so no separate large-file download is needed for quick verification. Appendix A.2.1 gives the proof-to-code correspondence, and Appendix A.2.3 states the trusted base and reproduction interface.

Fixed-width arithmetic. The final proof-relevant decisions have an independent arbitrary-precision replay. The eight-column square-matrix verifier reports maximum absolute determinant 56 and maximum absolute Cramer numerator 32. The paper does not treat SHA–256 as a mathematical inference; hashes bind the released source and records to the audited execution.

Search versus checking. The search programs may be heuristic and may choose different valid one-deficit roundings on different platforms. Byte identity of a newly proposed certificate file is not required. What is required is zero failures, exact record-set equality, and successful independent exact replay. A mismatch, omitted record, duplicate record, nonpositive exact margin, or nonempty failure file causes verification to terminate unsuccessfully.

A.2.1 Proof-to-code correspondence

The table gives the stable logical map. Operational build details remain in the artifact documentation so that compiler flags and paths can be updated without changing the mathematical proof.

Proof obligation	Authoritative source	Proof-relevant data or verdict
Six-row matrix-enumeration completeness	`src/common/enumerate_antichains.cpp`	`data/n6/antichains.txt`; 2,153 records
Six-row floor completeness	`src/common/generate_minimal_holes.cpp`, `src/common/generate_upper_cells.cpp`	32,286 minimal targets; 597 generated floor vectors
Six-row exact classification	both `src/common/classify_cells_*.cpp`	`data/n6/final_cells.txt`; empty counterexample outputs
Seven-row matrix-enumeration completeness	`src/common/enumerate_antichains.cpp`, `src/n7/enumerate_positive_dual.cpp`	`data/n7/positive_antichains_*.txt`; 61,846 records
Seven-row exact classification	both common classifiers	`data/n7/final_cells.txt`; 298 all-surplus regions; no counterexample records
Seven-row one-deficit integrity	`src/n7/verify_puncture_certificates.py`	`data/n7/puncture_certificates.json`
Seven-row universal separation	`src/n7/verify_square_surplus.cpp`, `src/n7/verify_rectangular_surplus.cpp`	294 regions from square matrices and four from rectangular matrices
Eight-row matrix hierarchy	`src/n8/m8_canonical_enumerator.cpp`, `src/n8/m8_kernel_list_verifier.cpp`	matrices for $m = 4, 5, 6, 7, 8$; exact square-matrix audit
Eight-row floor and unresolved-region scan	`src/n8/m8_floor_cell_scanner.cpp`, `src/n8/eight_row_floor_cell_scanner_template.cpp`	complete records for unresolved regions by m and κ
Eight-row certificate replay	`src/n8/m6_exact_certificate_checker.cpp`, `src/n8/exact_cps_fullsat_checker.cpp`, `src/n8/m8_exact_checker_ll.cpp`, `src/n8/m8_exact_checker_gmp.cpp`	1,085,459 fixed and 24 adaptive certificates in total

Proof obligation	Authoritative source	Proof-relevant data or verdict
Eight-row coverage	`src/n8/m8_record_checker.cpp`	unresolved regions equal certificates plus zero failures
Package integrity	`scripts/verify_manifest.py`	package-wide `SHA256SUMS`
Fast predecessor replay	`scripts/verify_predecessors.sh`	exact six- and seven-row `PASS` lines
Complete quick replay	`scripts/quick_verify.sh`	manifest, predecessor checks, eight-row replay, final `PASS`

A.2.2 Principal eight-column hashes

The proof-complete release records the following principal SHA–256 values:

```
4bff2f6e4af42bb2ff8517e08fd7ceff36767f8a6a7bd4b95f972799a7f597d0
  data/n8/m8/n8m8_pos.bin
f22c6d2ee7f8359e99ac84030a27f17588ae46c069831b57cd7f0f5f97d77ff4
  data/n8/m8/n8m8_hard.bin
527d8237a2e4aaa79a5767fc5b6cfed8e9094f11e62f413e49a023dff683021b
  data/n8/m8/cps8_all.bin
af5570f5a1810b7af78caf4bc70a660f0df51e42baf91d4de5b2328de0e83dfc
  data/n8/m8/cps8_fail.bin
```

The certificate hash identifies the exact-valid stream deposited with this release. A previous exact-valid proposal stream had a different byte hash; certificate byte identity is not a proof obligation. The last file is the zero-count failure header. The package-wide manifest binds every source, data, log, and summary file, not only these four principal records.

A.2.3 Trusted base and reproduction interface

The mathematical proof trusts the analytic reductions in this paper, the exact source identified above, a conforming compiler and exact-arithmetic libraries, and ordinary hardware execution. It does not trust floating-point feasibility or optimality verdicts. A search program may choose a different valid one-deficit rounding on another platform; byte identity of a newly proposed certificate file is unnecessary. What is required is complete coverage of the unresolved regions, zero failures, and successful independent exact replay.

The released artifact exposes

```
scripts/quick_verify.sh
```

for a referee-scale audit and

```
scripts/full_regenerate.sh /path/to/new/all-modules
```

for regeneration from canonical supports. The first command verifies the package manifest, recompiles the authoritative checkers, checks the six- and seven-row classifications, verifies every stored eight-row record set, and replays every final certificate. The second command regenerates canonical matrices, minimal nonimplementable floors, retained regions, records for unresolved regions, certificate proposals, exact outputs, and machine-readable summaries. Both commands terminate unsuccessfully on an omitted key, duplicate key, nonpositive exact margin, nonempty failure file, or count mismatch.

The packaged referee-scale run is recorded in `docs/FULL_M8_COMPLETION_REPORT.md` and `logs/n8/proof_complete_quick_verify.log`. The exact-valid certificate stream and its proposal portability are documented in `docs/CERTIFICATE_STREAM_PORTABILITY.md`.

A.3 Independent six-voter cross-check

An earlier, logically independent six-voter proof uses a different one-deficit price lemma and the five-row integralizability theorem. It reduces the regular six-voter case to 3,209 matrix–budget cases and verifies the relevant integer targets using two short integer-only C++ programs. This alternative proof is not used in the unified chain above, whose six-voter base is Computer Lemma 2.5. The older computation is distributed in `optional_cross_checks/independent_six_voter_proof` as an optional independent cross-check.